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MATRIX LOCALIZATIONS OF n-FIRS. II

BY

PETER MALCOLMSON

ABSTRACT. In a previous paper by this author and with a similar title, it was shown that adjoining universal inverses for all $k \times k$ full matrices over an n-fir results in the localized ring being an $(n - 2k)$-fir. In this note a counterexample is used to show that the result is best possible in general. Techniques of the previous paper are strengthened and a result on a kind of finite inertia of certain rings within their localizations is obtained.

In this note we extend the techniques of a previous paper **[4]** to complete the arguments regarding a counterexample given there. In that paper it was shown that if Σ consists of all $k \times k$ full matrices over an n-fir R (with $2k < n$), then the universal Σ-inverting ring R_Σ is an $(n - 2k)$-fir. A counterexample was suggested (but not proved) to show that this result was best possible—that is, R_Σ need not be an $(n - 2k + 1)$-fir. New arguments now allow us to prove this for the counterexample originally presented. They also can be used to give a kind of "finite" inertia of a ring R inside its matrix localization R_Σ.

We keep all the definitions from **[4]**, including the specialized one we now recall. A set Σ of square matrices over R is said to be *p-complete* if the following trivialization condition holds: for every matrix relation of the form

$$\begin{pmatrix} A & X & P \\ S & T & U \end{pmatrix} \begin{pmatrix} Q & V \\ G & W \\ B & Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where A and B are either null (0×0) or in Σ, and where X has $\leqslant p$ columns, there exists an invertible matrix Z over R such that

$$\begin{pmatrix} A & X & P \\ S & T & U \end{pmatrix} Z = \begin{pmatrix} A' & X' & 0 \\ S' & T' & 0 \end{pmatrix}$$

and

$$Z^{-1} \begin{pmatrix} Q & V \\ G & W \\ B & Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ G' & W' \\ B' & Y' \end{pmatrix},$$

where A' and B' are either null or in Σ, and where

$$\begin{pmatrix} X' \\ T' \end{pmatrix} \begin{pmatrix} G' & W' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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is a trivial relation (i.e., each column in $\binom{X'}{T'}$ is either zero or the corresponding row in $(G' \quad W')$ is zero).

The results of [**4**] include the facts that if Σ is p-complete, then it consists of non-zero-divisors and its multiplicative closure $\overline{\Sigma}$ is also p-complete. The first result of this note is the following extension of Lemma 2 of [**4**].

LEMMA 2^+. *Let Σ be 0-complete, $\overline{\Sigma}$ its multiplicative closure and $\lambda\colon R \to R_\Sigma$ the universal Σ-inverting homomorphism. Then an $i \times j$ matrix N over R has $\lambda(N) = \lambda(F)\lambda(A)^{-1}\lambda(X)$ (where $A \in \overline{\Sigma}$ and F, X are appropriately sized) in R_Σ if and only if there exist P, Q of the same shape as A (i.e., same diagonal block sizes), with $P \in \overline{\Sigma}$ and U, V, all over R, such that*

$$\begin{pmatrix} A & X \\ F & N \end{pmatrix} = \begin{pmatrix} P \\ U \end{pmatrix} (Q \quad V).$$

PROOF. Lemma 2 of [**4**] is the same statement for $i = j = 1$, so we proceed by induction on $i + j > 2$. If $j \geqslant 2$, rewrite X as $(X_1 \quad X_2)$ and N as $(N_1 \quad N_2)$. By induction,

$$\begin{pmatrix} A & X_1 \\ F & N_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & X_2 \\ F & N_2 \end{pmatrix}$$

both factor as before, say

$$\begin{pmatrix} A & X_1 \\ F & N_1 \end{pmatrix} = \begin{pmatrix} P_1 \\ U_1 \end{pmatrix} (Q_1 \quad V_1),$$
$$\begin{pmatrix} A & X_2 \\ F & N_2 \end{pmatrix} = \begin{pmatrix} P_2 \\ U_2 \end{pmatrix} (Q_2 \quad V_2),$$

where $P_1, P_2 \in \overline{\Sigma}$.

We now need to use Lemma 3 of [**4**] (or some symmetry), which states that if Σ is 0-complete and if $A = BC$ with $A, B \in \Sigma$, then there is an invertible E with $EC \in \Sigma$. Applying this to $A = P_2 Q_2$ we see that we can get E' invertible with $E'Q_2 = Q_2' \in \overline{\Sigma}$, still the same shape as A. Now we let $P_2' = P_2 E'^{-1}$, $U_2' = U_2 E'^{-1}$, $V_2' = E'V_2$, so that

$$\begin{pmatrix} A & X_2 \\ F & N_2 \end{pmatrix} = \begin{pmatrix} P_2' \\ U_2' \end{pmatrix} (Q_2' \quad V_2').$$

We now see that

$$\begin{pmatrix} P_1 & -P_2' \\ U_1' & -U_2' \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with $P_1, Q_2' \in \overline{\Sigma}$. Using 0-completeness, there exists an invertible Z such that

$$\begin{pmatrix} P_1 & -P_2' \\ U_1 & -U_2' \end{pmatrix} Z = \begin{pmatrix} A_1 & 0 \\ S_1 & 0 \end{pmatrix} \quad \text{and} \quad Z^{-1} \begin{pmatrix} Q_1 \\ Q_2' \end{pmatrix} = \begin{pmatrix} 0 \\ B_2 \end{pmatrix},$$

say, where $A_1, B_2 \in \bar{\Sigma}$, all the same shape. Then

$$\begin{pmatrix} A_1 \\ S_1 \end{pmatrix}(I \quad 0)Z^{-1}\begin{pmatrix} Q_1 & V_1 & 0 \\ 0 & 0 & -V_2' \end{pmatrix} = \begin{pmatrix} P_1 & -P_2' \\ U_1 & -U_2' \end{pmatrix}\begin{pmatrix} Q_1 & V_1 & 0 \\ 0 & 0 & -V_2' \end{pmatrix}$$
$$= \begin{pmatrix} A & X_1 & X_2 \\ F & N_1 & N_2 \end{pmatrix} = \begin{pmatrix} A & X \\ F & N \end{pmatrix},$$

providing the required factorization (the first block in the second factor has the same shape as A by an examination of the block diagonal structure, just as in the proof of Lemma 2). If $j = 1$ then a symmetrical argument will work to complete the induction.

Now we will use the extended Lemma 2^+ to prove our main result, which is essentially a further generalization of the original lemma. Note here that A may be null.

THEOREM. *Let Σ be p-complete, $\bar{\Sigma}$ its multiplicative closure, and λ: $R \to R_\Sigma$ the universal Σ-inverting homomorphism. Suppose $A \in \bar{\Sigma}$ is $n \times n$ $(n \geq 0)$ and let $m \leq p$. Then $\lambda(N) - \lambda(F)\lambda(A)^{-1}\lambda(X)$ has inner rank $\leq m$ over R_Σ (where F, X, N are of appropriate sizes over R) if and only if there is a factorization*

$$\begin{pmatrix} A & X \\ F & N \end{pmatrix} = \begin{pmatrix} P & J \\ U & K \end{pmatrix}\begin{pmatrix} Q & V \\ L & M \end{pmatrix} \quad \text{over } R,$$

where $P \in \bar{\Sigma}$ has the same shape as A and where the number of columns in J is $\leq m$. Hence, if $\lambda(N) - \lambda(F)\lambda(A)^{-1}\lambda(X)$ has inner rank $\leq m$, then $\left(\begin{smallmatrix} A & X \\ F & N \end{smallmatrix}\right)$ has inner rank $\leq n + m$ over R.

PROOF. For the purposes of the proof we will suppress the notation λ, since it is an injection anyway (Lemma 2 of [**4**]). If such a factorization occurs over R, then calculation shows

$$N - FA^{-1}X = (K - UP^{-1}J)(M - LA^{-1}X),$$

so the inner rank is $\leq m$ over R_Σ.

To prove the converse, assume the inner rank of $N - FA^{-1}X$ is $\leq m$, so $N - FA^{-1}X$ may be written as a product of a matrix with m columns over R_Σ times one with m rows. As in the proof of Theorem 1 of [**4**], we may assume the matrices over R_Σ are of the forms $GB^{-1}Y$ and $HC^{-1}Z$, where G, H, Y, Z are over R and $B, C \in \bar{\Sigma}$. Then from $N - FA^{-1}X = GB^{-1}YHC^{-1}Z$ we may calculate that

$$N = (F \quad G \quad 0 \quad 0)\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & -Y & 0 \\ 0 & 0 & I & -H \\ 0 & 0 & 0 & C \end{pmatrix}^{-1}\begin{pmatrix} x \\ 0 \\ 0 \\ Z \end{pmatrix}$$

over R_Σ, where the identity matrix I is $m \times m$. By using Lemma 2^+ and the 0-completeness of $\overline{\Sigma}$ we get a factorization over R,

(1)

$$\begin{pmatrix} A & 0 & 0 & 0 & X \\ 0 & B & -Y & 0 & 0 \\ 0 & 0 & I & -H & 0 \\ 0 & 0 & 0 & C & Z \\ F & G & 0 & 0 & N \end{pmatrix} \begin{pmatrix} A_1 & * & * & * \\ 0 & B_1 & * & * \\ 0 & 0 & D_1 & * \\ 0 & 0 & 0 & C_1 \\ * & * & * & * \end{pmatrix} \begin{pmatrix} A_2 & * & * & * & * \\ 0 & B_2 & * & * & * \\ 0 & 0 & D_2 & * & * \\ 0 & 0 & 0 & C_2 & * \end{pmatrix},$$

where $A_1,\ B_1,\ D_1,\ C_1 \in \overline{\Sigma}$, A_1 has the same shape as A, and where the asterisks represent unlabelled blocks.

We can use Lemma 3 of **[4]** as before to get that $EC_2 \in \overline{\Sigma}$ for some invertible E over R. Then we can multiply the left factor in (1) by

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & E^{-1} \end{pmatrix}$$

on the right, and the right factor by the inverse of this matrix; or, equivalently, we may assume in (1) that $A_1,\ B_1,\ C_2 \in \overline{\Sigma}$.

Now consider the product of the second block row times the last two block columns in (1):

$$(0 \quad B_1 \quad \boxed{*} \quad *) \begin{pmatrix} * & * \\ * & * \\ \hline \multicolumn{1}{|c}{*} & \multicolumn{1}{c|}{*} \\ \hline C_2 & * \end{pmatrix} = (0 \quad 0). \tag{2}$$

We may use p-completeness of $\overline{\Sigma}$ here, since the outlined block in the first factor has $m \leqslant p$ columns. The invertible matrix obtained will alter only the last three blocks of each factor, giving a new form of (2):

$$(0 \quad B_1' \quad \boxed{* \quad 0} \quad 0) \begin{pmatrix} * & * \\ 0 & 0 \\ \hline \multicolumn{1}{|c}{0} & \multicolumn{1}{c|}{0} \\ \multicolumn{1}{|c}{*} & \multicolumn{1}{c|}{*} \\ \hline C_2' & * \end{pmatrix} = (0 \quad 0),$$

where $B_1',\ C_2' \in \overline{\Sigma}$ and where the now-trivialized outlined blocks have been explicitly divided into their zero/nonzero rows/columns. The effect of the invertible matrix on the original factorization (1) changes it into

$$\left(\begin{array}{cc|cc|c} A_1 & * & * & * & * \\ 0 & B_1 & * & 0 & 0 \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ * & * & * & * & * \end{array}\right) \left(\begin{array}{ccccc} A_2 & * & * & * & * \\ 0 & * & * & 0 & 0 \\ \hline 0 & * & * & 0 & 0 \\ 0 & * & * & * & * \\ \hline 0 & * & * & C_2' & * \end{array}\right). \tag{3}$$

Now we consider the product of the first and last block rows times the next-to-last block column,

$$\begin{pmatrix} A_1 & * & \boxed{\begin{matrix} * & * \\ * & * \end{matrix}} & * \\ * & * & & * \end{pmatrix} \begin{pmatrix} * \\ 0 \\ \boxed{\begin{matrix} 0 \\ * \end{matrix}} \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Again we may use p-completeness of $\overline{\Sigma}$, obtaining an invertible matrix which only affects the first one and last two block columns of the left factor (and similarly on the right.) We get a new form of the above, namely

$$\begin{pmatrix} P & * & \boxed{\begin{matrix} * & J & 0 \\ * & K & 0 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ U & * & & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \boxed{\begin{matrix} 0 \\ 0 \\ * \end{matrix}} \\ C_2'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where P, $C_2'' \in \overline{\Sigma}$ and P has the same shape as A. Again we have explicitly divided the now-trivialized outlined blocks, and we have also labelled certain of the blocks. The effect on the entire factorization (3) changes it into

$$\begin{pmatrix} P & * & * & J & 0 & 0 \\ * & B' & * & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ U & * & * & K & 0 & 0 \end{pmatrix} \begin{pmatrix} Q & * & * & 0 & V \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ L & * & * & 0 & M \\ * & * & * & * & * \\ * & * & * & C_2'' & * \end{pmatrix}$$

$$= \begin{pmatrix} A & 0 & 0 & 0 & X \\ 0 & B & -Y & 0 & 0 \\ 0 & 0 & I & -H & 0 \\ 0 & 0 & 0 & C & Z \\ F & G & 0 & 0 & N \end{pmatrix},$$

where again we have labelled certain of the blocks. Now certainly

$$\begin{pmatrix} A & X \\ F & N \end{pmatrix} = \begin{pmatrix} P & J \\ U & K \end{pmatrix} \begin{pmatrix} Q & V \\ L & M \end{pmatrix}$$

and J has $\leqslant m$ columns, as desired.

From the theorem we get an immediate corollary on preserving inner ranks.

COROLLARY A. *Let Σ be p-complete, and λ: $R \to R_\Sigma$ the universal homomorphism. Let N be any matrix over R with inner rank m and let m_Σ be the inner rank of its image over R_Σ. Then always $m_\Sigma \leqslant m$, and if $m_\Sigma < m$ then $p < m_\Sigma$. In particular, λ maps full $(p+1) \times (p+1)$ matrices to full matrices.*

Now let us proceed to the counterexample suggested in [**4**]. Let F be a field and let L be the F-algebra generated by two (noncommuting) variables c, d, satisfying $cd = 0$. The ring $W_n(L)$ is the F-algebra generated by the entries of two $n \times n$ matrices C, D of variables satisfying $CD = 0$. Bergman [**1**] has shown that $W_n(L)$ is an $(n - 1)$-fir but not an n-fir.

COROLLARY B. *Let L be as above, $0 < k < n/2$ and let $R = W_{n+1}(L)$, an n-fir. Let Σ consist of all $k \times k$ full matrices over R. Then the universal Σ-inverting ring R_Σ is an $(n - 2k)$-fir but not an $(n - 2k + 1)$-fir.*

PROOF. Here we suppress the notation for $\lambda\colon R \to R_\Sigma$ as before, and we regard $R \subset R_\Sigma$. We already know that R_Σ is an $(n - 2k)$-fir by Corollary 1 of [**4**]. Let $p = n - 2k + 1$ and recall the matrices C, D over R with $CD = 0$. Write the first $k + p$ rows of C as

$$\begin{pmatrix} A & X & P \\ S & T & U \end{pmatrix},$$

where A is $k \times k$, X is $k \times p$, etc. Similarly write the first $k + p$ columns of D as

$$\begin{pmatrix} Q & V \\ G & W \\ B & Y \end{pmatrix},$$

where B is $k \times k$, G is $p \times k$, etc. Then

$$\begin{pmatrix} A & X & P \\ S & T & U \end{pmatrix}\begin{pmatrix} Q & V \\ G & W \\ B & Y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, by mapping C to I and D to 0 over F we can see that A must be full over R (and similarly B is full). Then $A, B \in \Sigma$ and by a calculation as in [**4**] we see that

$$(T - SA^{-1}X)(W - GB^{-1}Y) = 0$$

over R_Σ, where both factors are $p \times p$.

Now suppose R_Σ is a p-fir. Then by trivialization we could see that the sum of the inner ranks of $T - SA^{-1}X$ and $W - GB^{-1}Y$ over R_Σ is some $m \leqslant p$ (the Sylvester inequality; see [**3**]). By the Theorem of this paper we must have that the sum of the inner ranks of

$$\begin{pmatrix} A & X \\ S & T \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} G & W \\ B & Y \end{pmatrix}$$

over R is $\leqslant 2k + m \leqslant 2k + p$. But as before we can establish that both these matrices are full over R (as submatrices of C, D), so their ranks sum to $2k + 2p > 2k + p$. This contradiction shows R_Σ is not an $(n - 2k + 1)$-fir.

This result, together with the remarks of [**4**] for $p = 1$, completes the presentation of the counterexample. We may remark that the above construction results in a pair of $p \times p$ full matrices over R_Σ (use the Theorem) whose product is zero. This suggests that we may give a definition of *deficiency* of a $(p - 1)$-fir R: with each pair of an $i \times p$ matrix C and a $p \times j$ matrix D such that $CD = 0$, we associate the number which is the sum of the inner ranks of C and D. If m is the maximum such number among all such pairs, then the deficiency $\mathrm{dp}(R) = m - p$. Under this definition a p-fir would have deficiency 0, while the R_Σ of Corollary B would have the maximum deficiency p.

Before describing the sort of "finite" inertia that may hold in our situation, we recall the actual definitions from [**2**]. Let R be a subring of a ring S. Given a row vector X and a column vector Y over S of the same length and such that $XY \in R$, we shall say that XY lies *trivially* in R if for each entry of X one of the following holds: (1) the entry is zero; (2) the corresponding entry in Y is zero; or (3) both corresponding entries in X and Y lie in R.

Then R will be said to be *n-inert* in S if the following condition holds: whenever we have any family $\{X_\lambda\}$ of rows over S with n entries and any family $\{Y_\mu\}$ of columns over S with n entries, such that $X_\lambda Y_\mu \in R$ for every λ, μ, then there exists an invertible $n \times n$ matrix P over S such that every product $(X_\lambda P)(P^{-1}Y_\mu)$ lies trivially in R.

We wish to adjust this definition (as suggested by Proposition 1.6.2 of [**2**]) for our purposes: we will say that R is *finitely n-inert* in S if the condition above is required to hold for all finite families $\{X_\lambda\}$, $\{Y_\mu\}$. It can be seen (as in [**2**]) that this is equivalent to the following condition: given two matrices X, Y over S such that X has n columns, Y has n rows and XY is over R, there exists an $n \times n$ invertible matrix P over S such that XP has the block form $(X_1 \quad X_0 \quad 0)$ and $P^{-1}Y$ has the compatible block form

$$\begin{pmatrix} 0 \\ Y_0 \\ Y_2 \end{pmatrix},$$

where X_0, Y_0 are over R.

PROPOSITION. *Let Σ be p-complete and let λ: $R \to R_\Sigma$ be the universal homomorphism. Then $\lambda(R)$ is finitely p-inert in R_Σ.*

PROOF. Again we suppress the notation λ. As before, instead of X over R_Σ we may write $FA^{-1}X$, where F, X are over R, A is in the multiplicative closure of Σ, and X has p columns. Instead of Y we write $GB^{-1}Y$, where $B \in \overline{\Sigma}$, G, Y over R, and G having p rows. The assumption is then $(FA^{-1}X)(GB^{-1}Y) = N$, a matrix over R. By Lemma 2^+ we get a factorization as follows:

$$\begin{pmatrix} A & -XG & 0 \\ 0 & B & Y \\ F & 0 & N \end{pmatrix} = \begin{pmatrix} A_1 & P \\ 0 & B_1 \\ S & T \end{pmatrix} \begin{pmatrix} A_2 & Q & V \\ 0 & B_2 & W \end{pmatrix},$$

where by the standard juggling we may assume $A_1, B_2 \in \overline{\Sigma}$.

Now consider the product

$$\begin{pmatrix} A_2 & X & P \\ S & 0 & T \end{pmatrix} \begin{pmatrix} Q & V \\ G & 0 \\ B_2 & W \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}.$$

We may trivialize (using p-completeness) the portion

$$(A_1 \quad X \quad P)\begin{pmatrix} Q & V \\ G & 0 \\ B_2 & W \end{pmatrix} \quad \text{into} \quad (A_1' \quad X' \quad 0)\begin{pmatrix} 0 & 0 \\ G' & V' \\ B_2' & W' \end{pmatrix} = (0 \quad 0),$$

where the relation $X'(G' \quad V'') = (0 \quad 0)$ is trivial. The effect on the original product changes it into

$$\begin{pmatrix} A_1' & X' & 0 \\ S' & T' & U' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ G' & V' \\ B_2' & W' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}.$$

Now we may trivialize the portion

$$(T' \quad U')\begin{pmatrix} G' \\ B_2' \end{pmatrix} = 0,$$

adjusting only the nonzero rows of G', B_2' (and thus not altering A_1' or X_1'). The effect on the product is then

$$\begin{pmatrix} A_1' & X' & 0 \\ S' & T'' & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ G'' & V'' \\ B_2'' & W'' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix},$$

where *both* relations $X'(G'' \quad V'') = (0 \quad 0)$ and

$$\begin{pmatrix} X' \\ T'' \end{pmatrix} G'' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are trivial. (Note that now N can be seen to have inner rank $\leqslant p$ over $\mathbf{R}$.)

The result of the two trivializations is that there exists an invertible matrix Z over R such that

$$\begin{pmatrix} A_1 & X & P \\ S & 0 & T \end{pmatrix} Z = \begin{pmatrix} A_1' & X' & 0 \\ S' & T'' & 0 \end{pmatrix} \quad \text{and} \quad Z^{-1}\begin{pmatrix} Q & V \\ G & 0 \\ B_2 & W \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ G'' & V'' \\ B_2'' & W'' \end{pmatrix},$$

where $A_1, A_1', B_2, B_2'' \in \overline{\Sigma}$, and where both $X'(G'' \quad V'')$ and $T''G''$ are trivially zero. Now, just as in the proof of Theorem 1 of **[4]** we can see that

$$\underline{D} = \begin{pmatrix} 0 & I & -GB_2^{-1} \end{pmatrix} Z \begin{pmatrix} -A_1'^{-1}X' \\ I \\ 0 \end{pmatrix}$$

is an invertible matrix over R_Σ. Computation as before will then show that

$$\begin{aligned} (FA^{-1}X)\underline{D} &= (-S \quad 0 \quad -T) Z \begin{pmatrix} -A_1'^{-1}X' \\ I \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} S'A_1'^{-1} & -I \end{pmatrix}\begin{pmatrix} X' \\ T'' \end{pmatrix}. \end{aligned}$$

Similarly,

$$\underline{D}^{-1}(GB^{-1}Y) = (G'' \quad V'')\begin{pmatrix} B_2''^{-1}W'' \\ -I \end{pmatrix}.$$

Let us write

$$\begin{pmatrix} X' \\ T'' \end{pmatrix} = \begin{pmatrix} X_1' & 0 & 0 \\ T_1'' & T_2'' & 0 \end{pmatrix} \quad \text{and} \quad (G'' \quad V'') = \begin{pmatrix} 0 & 0 \\ 0 & V_2'' \\ G_3'' & V_3'' \end{pmatrix},$$

recalling that these are already trivial relations. Then

$$FA^{-1}X\underline{D} = \begin{pmatrix} S'A_1'^{-1} & -I \end{pmatrix} \begin{pmatrix} X_1' & 0 & 0 \\ T_1'' & T_2'' & 0 \end{pmatrix} = \begin{pmatrix} S'A_1'^{-1}X_1 - T_1'' & -T_2'' & 0 \end{pmatrix}$$

and, similarly,

$$\underline{D}^{-1}GB^{-1}Y = \begin{pmatrix} 0 & 0 \\ 0 & V_2'' \\ G_3'' & V_3'' \end{pmatrix} \begin{pmatrix} B_2''^{-1}W'' \\ -I \end{pmatrix} = \begin{pmatrix} 0 \\ -V_2'' \\ G_3''B_2''^{-1}W'' - V_3'' \end{pmatrix}.$$

Thus $(FA^{-1}X\underline{D})(\underline{D}^{-1}GB^{-1}Y) = N$ lies trivially in R, as claimed.

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DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202